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All higher genus BPS membranes in the plane wave background

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Abstract: We consider supermembranes in the maximally supersymmetric plane wave geometry of the eleven dimensions and construct complete solutions of the continuum version of the 1/4 BPS equations. The supermembranes may have an arbitrary number of holes and arbitrary cross sectional shapes. In the matrix regularized version, we solve the matrix equations for several simple cases including fuzzy torus. In addition, we show that these solutions are trivially generalized to 1/8 and 1/16 BPS configuations.

Keywords: Solitons Monopoles and Instantons, Penrose limit and pp-wave background, M(atrix) Theories.

Contents

1. Introduction

The 1/2 BPS supermembrane and their matrix regularized version in the maximally supersymmetric plane wave geometry of the eleven dimensions are first introduced in reference [1]. Since then the matrix model have been studied in its various aspects [2, 10].

In this note, we like to answer the following simple question. In reference [2], the 1/4 BPS equations involving 1,2,3 directions are found. They are just one mass parameter deformation of the 1/4 BPS equations [11, 12] of supertubes [13, 11] in the ordinary BFSS matrix model [14]. The shape of the supertubes involves arbitrariness; the cross sectional shape may be completely arbitrary while preserving a quarter of supersymmetries [15, 16]. Matrix solutions of arbitrary cross sectional shapes were constructed explicitly in reference [17]. The mass deformation is expected not to ruin the arbitrariness of supertube solutions. Thus the question is whether one can identify the corresponding arbitrariness of supermembranes in the plane wave geometry.

We shall show that this is indeed possible by constructing the most general solutions of the mass deformed BPS equations. For the continuum version prior to the matrix regularization, we are able to construct the complete solutions of the BPS equations. The surfaces may have an arbitrary number of holes and the cross sectional shapes. Further, the surfaces may be either compact or noncompact. For the matrix BPS equations, finding general solution of the matrix BPS equation seems very complicated. Hence we consider only the limited cases of genus one surfaces corresponding to the fuzzy torus [4].

As in the case of supertubes, the general solutions may be used to identify the degeneracy of configurations for given charges. One needs to fix the energy, angular momentum and the charges and counts the corresponding degeneracy of the systems [18, 19]. This may then be compared with the entropy of the corresponding charged supersymmetric black holes as in the flat case [20 − 22]. Such black hole solution asymptotic to the plane wave geometry is not known and its construction will be very interesting. See reference [23] for some related attempt to construct intersecting brane solutions asymptotic to the plane wave geometry.

In section 2, we review the membrane actions and the 1/4 BPS equations. We construct the complete solutions of the BPS equations in section 3. Section 4 deals with the matrix solutions. The last section comprises the concluding remarks.

2. Membranes in the plane wave geometry

Our mission is to study the dynamics of M2 branes in the maximally supersymmetric plane wave background,

$$\begin{aligned} ds^2 &= -4dx^-dx^+ - \left(\frac{\mu}{6}\right)^2 \left(4\sum_{i=1}^{3} x_i^2 + \sum_{a=4}^{9} x_a^2\right)(dx^+)^2 + dx^I dx^I \\ F_{+123} &= \mu \end{aligned} \tag{2.1}$$

where the capital indices run over $1, 2, \cdots, 9$. This geometry can be obtained from the $AdS_7 \times S^4$ by taking the Penrose limit. The $1, 2, 3$ directions are from coordinates of S^4 and the remaining six directions are related to the spatial directions of the AdS_7 while one of the angular directions in S^4 is used in the light cone coordinate. The membrane action [24, 25] for the general background is given by the Nambu-Gotto action plus the Wess-Zumino action. By the standard procedure [24, 25], this action may be rewritten in the light cone frame as

$$L = L_0 + L_\mu \tag{2.2}$$

with

$$\begin{aligned} L_0 &= \frac{1}{2}\int d^2\sigma \left(p^+ \sum_I (D_0 X_I)^2 - \frac{1}{p^+}\sum_{I<J}\{X_I, X_J\}^2 + i\psi^T D_0 \psi + \frac{i}{p^+}\sum_I \psi^T \gamma_I \{X_I, \psi\}\right) \\ L_\mu &= -\frac{1}{2}\int d^2\sigma \Bigg(\mu^2 p^+ \left(\frac{1}{9}\sum_{i=1}^{3} X_i^2 + \frac{1}{36}\sum_{a=4}^{9} X_a^2\right) - \\ &\qquad - \frac{i\mu}{3}\sum_{i,j,k=1}^{3} \{X_i, X_j\} X_k \epsilon_{ijk} + \frac{i\mu}{4}\psi^T \gamma_{123}\psi \Bigg), \end{aligned} \tag{2.3}$$

where p^+ is the conserved momentum related to the x^- translation, $x^+ = t$ is used as the light-cone time coordinate and the Poisson bracket is defined by

$$\{A, B\} = \frac{\partial A}{\partial \sigma_1}\frac{\partial B}{\partial \sigma_2} - \frac{\partial A}{\partial \sigma_2}\frac{\partial B}{\partial \sigma_1}.$$

The 16 dimensional Majonara fermions are used for the fermionic part and the gamma matrices are taken to be real. One comment is that, when the spatial section of M2 brane is compact, we call it as (an expanded) giant graviton [26] because the quantum numbers are matching. The Gauss law constraint here follows from setting the induced world volume metric

$$g_{t\alpha} = \frac{\partial X^M}{\partial t}\frac{\partial X^N}{\partial \sigma^\alpha} G_{MN} = 0, \tag{2.4}$$

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which is achieved by an appropriate worldvolume coordinate transformation. Using the metric in (2.1), one sees that the above condition becomes

$$4\frac{\partial}{\partial\sigma^\alpha}X^- = \sum_{I=1}^{9} D_t X^I \frac{\partial}{\partial\sigma^\alpha}X^I\,. \tag{2.5}$$

Application of $\epsilon^{\beta\alpha}\partial/\partial\sigma^\alpha$ on the both sides leads to the Gauss law constraint. Thus the Gauss law is simply the integrability condition of (2.5).

The matrix regularization is useful for the quantum mechanical description; one replaces the functions on the M2 brane worldvolume by hermitian $N\times N$ matrices and the Poisson brackets by commutators and so on. Namely the rules are summarized by

$$\begin{aligned} \{\ ,\ \} &\longleftrightarrow -iN^2[\ ,\ ] \\ \int d^2\sigma &\longleftrightarrow N\ \mathrm{tr} \\ X_I(\sigma) &\longleftrightarrow \frac{1}{N}X_I \\ \psi(\sigma) &\longleftrightarrow \frac{1}{\sqrt{N}}\psi\,. \end{aligned} \tag{2.6}$$

In addition, we introduce a new length scale R by N/p^+, which is identified with the radius of x^- circle.

The resulting matrix model lagrangian [1] becomes

$$\tilde{L} = \tilde{L}_0 + \tilde{L}_\mu \tag{2.7}$$

with

$$\begin{aligned} 2\tilde{L}_0 &= \frac{1}{R}\,\mathrm{tr}\left(\sum_I (D_0X_I)^2 + R^2\sum_{I<J}[X_I,X_J]^2\right) + \mathrm{tr}\left(i\psi^T D_0\psi + \sum_I R\psi^T\gamma_I[X_I,\psi]\right) \\ 2\tilde{L}_\mu &= -\frac{\mu^2}{R}\,\mathrm{tr}\left(\frac{1}{9}\sum_{i=1}^{3}X_i^2 + \frac{1}{36}\sum_{i=4}^{9}X_i^2\right) - \frac{2\mu i}{3}\sum_{i,j,k=1}^{3}\mathrm{tr}\,X_iX_jX_k\epsilon_{ijk} - \frac{\mu i}{4}\,\mathrm{tr}\,\psi^T\gamma_{123}\psi\,, \end{aligned} \tag{2.8}$$

The lagrangian L_0 is the same as the usual matrix model in [14]. L_μ includes mass terms and the coupling to the four form field strength background. This matrix model has been studied in various aspects [2, 10]. Under the supersymmetry, the fermionic field transforms as

$$\delta\psi = \left(D_0X_I - \frac{i}{2}[X_I,X_J]\gamma_{IJ} + \frac{\mu}{3}\sum_{i=1}^{3}X_i\gamma_i\gamma_{123} - \frac{\mu}{6}\sum_{i=4}^{9}X_i\gamma_i\gamma_{123}\right)\epsilon \tag{2.9}$$

where $\epsilon = e^{-\frac{1}{12}\gamma_{123}t}\epsilon_0$ with a constant spinor ϵ_0.

In this note, we like to find solution of the BPS equations constructed in reference[2]. Among many others, we like to focus first on the 1/4 BPS equations with preserved supersymmetries $P_\mp\epsilon$ with the projection operator $P_\pm = (1\pm\gamma_3)/2$. The BPS equations [2] read

$$i[X_1,X_2] + \frac{\mu}{3R}X_3 = 0\,, \qquad D_0X_3 = 0$$

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$$i[X_1, X_3] - \frac{\mu}{3R}X_2) \pm \frac{1}{R}D_0 X_1 = 0$$
$$i[X_2, X_3] + \frac{\mu}{3R}X_1 \pm \frac{1}{R}D_0 X_2 = 0\,. \tag{2.10}$$

In addition, one has to satisfy the Gauss law constraint,

$$[X_1, D_0 X_1] + [X_2, D_0 X_2] = 0\,. \tag{2.11}$$

For definiteness, we shall choose the + sign projection corresponding to the remaining supersymmetries of $P_- \epsilon$. The analysis below may be generalized to the other types of BPS equations involving other directions. Notice that these BPS equations may be alternatively derived by considering the bosonic part of the hamiltonian. By the technique of completing squares, one may show that the hamiltonian is bounded by conserved quantities,

$$H \geq \pm \left[C_F - \frac{\mu}{3} J_{12} \right], \tag{2.12}$$

where C_F is the energy from the fundamental string tension [27, 12],

$$C_F = i \ \mathrm{tr}\ \left([X_1, X_3 D_0 X_1] + [X_2, X_3 D_0 X_2] \right), \tag{2.13}$$

and the angular momentum J_{12} is defined by

$$J_{12} = \frac{1}{R}\ \mathrm{tr}(X_1 D_0 X_2 - X_2 D_0 X_1)\,. \tag{2.14}$$

For the compact surfaces of giant graviton, $C_F = 0$ because the representations are finite dimensional and the trace of commutators then vanish. (For the continuum description, the string charge of compact surface is again zero because it is given by integral of total derivative.)

The above BPS equation may be simplified by taking the gauge $A_0 = RX_3$; the last two equations of (2.10) reduce to

$$\dot{X}_1 + i\dot{X}_2 = -\frac{\mu}{3R} i(X_1 + iX_2)\,. \tag{2.15}$$

Its general solution is then given by

$$X_1 + iX_2 = e^{-\frac{\mu}{3R} it}(Y_1 + iY_2) \tag{2.16}$$

where Y_1 and Y_2 are real and time independent. The second equation of (2.10) implies that $X_3 \equiv Z$ is also constant in time. Using this solution, the full BPS equations including the Gauss law constraint become

$$[Y_1, [Y_1, Z]] + [Y_2, [Y_2, Z]] - 2\left(\frac{\mu}{3R}\right)^2 Z = 0\,, \qquad [Y_1, Y_2] = i\frac{\mu}{3R} Z\,. \tag{2.17}$$

These generalize the BPS equations associated with the matrix description of supertubes by a mass parameter [11, 12] as noticed in reference [2]. Note that the abelian or center of mass rotation along X_1, X_2 plane decouples from the nonabelian part without further breaking the supersymmetry.

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A given 1/4 BPS configuration can be excited further by rotating the abelian, or center of mass, part along the $X_4, X_5, ..., X_9$, while remaining BPS with less supersymmetries. For example, we can require that the the spinor ϵ_0 satisfying the 1/4 BPS condition $\gamma_3\epsilon_0 = \epsilon_0$ to satisfy an additional compatible condition $\gamma_{12345}\epsilon_0 = \epsilon_0$, which breaks the susy further to 1/8 and the BPS equation along X_4 and X_5 can be solved by $X_4 + X_5 = e^{-\frac{iu}{6}t}(Y_4 + iY_5)$ with constant matrices Y_4 and Y_5 proportional to the identity matrix. The 1/8 BPS configuration would carry nonzero angular momentum J_{45}. Similarly the additional compatiable requirement $\gamma_{12367}\epsilon_0 = \epsilon_0$ breaks susy to 1/16. The 1/16 BPS configuration satisfying $X_6+X_7 = e^{-\frac{iu}{6}t}(Y_6+iY_7)$ where Y_6, Y_7 are proportional to identity matrix would carry nonzero angular momentum J_{67}. Since $\gamma_{12...9} = 1$, the susy parameter ϵ_0 for the 1/16 BPS also satisfies also the constraint $\gamma_{12389}\epsilon_0 = \epsilon_0$. Thus the configuration whose abelian part satisfying $X_8 + X_9 = e^{-\frac{iu}{6}t}(Y_8 + iY_9)$ with constant abelian $Y_8, 9$ would remain 1/16 BPS. Indeed for such a 1/16 BPS configuration, the hamiltonian would become

$$H_{\frac{1}{16}} = C_F - \frac{\mu}{3}J_{12} - \frac{\mu}{6}(J_{45} + J_{67} + J_{89}) \tag{2.18}$$

which is exactly what to expect from the BPS states from the superalgebra.

Before closing this review, let us record the corresponding BPS equations in the form of the continuum description. Using the correspondence, the BPS equations may be put into the form of

$$\begin{aligned} \{X_1, X_2\} - \frac{\mu p^+}{3}X_3 &= 0\,, \qquad D_0X_3 = 0 \\ \{X_1, X_3\} + \frac{\mu p^+}{3}X_2 \mp p^+D_0X_1 &= 0 \\ \{X_2, X_3\}) - \frac{\mu p^+}{3}X_1 \mp p^+D_0X_2 &= 0\,, \end{aligned} \tag{2.19}$$

with the Gauss law constraint,

$$\{X_1, D_0X_1\} + \{X_2, D_0X_2\} = 0\,. \tag{2.20}$$

In a similar manner, these equations may be reduced to

$$\{Y_1, \{Y_1, Z\}\} + \{Y_2, \{Y_2, Z\}\} + 2\left(\frac{\mu p^+}{3}\right)^2 Z = 0\,, \qquad \{Y_1, Y_2\} = \frac{\mu p^+}{3}Z\,. \tag{2.21}$$

There are some known solutions of these matrix BPS equations. The simplest ones are the 1/2 BPS vacuum solutions of zero energy and angular momentum, which satisfies $D_0X_I = 0$ and so satisfies the SU(2) algebra and can be represented as fuzzy spheres. The 1/4 BPS rotating ellipsoidal branes [2] has non zero energy and angular momentum and the corresponding configurations are constructed again based on the SU(2) algebra. The nature of the full solutions has not been worked out.

There appeared quite general solutions of the above continuum equations [4]. (Here we shall use a completely different method from that of [4].) But the precise nature and implication does not seem to be understood. As discussed in the introduction, there are some hints from the analysis of supertubes; there should be a huge family of solutions because the above BPS equations generalize the BPS equation for the matrix description of supertubes and it is now well established that supertubes allow the arbitrary cross sectional shapes. As the BPS configurations with less supersymmetry could build from the 1/4 BPS configurations by giving the center of mass motion, there would be no additional degeneracy. Thus we just focus on 1/4 BPS configurations.

Our main purpose in this note is to explore the corresponding arbitrariness by constructing generic supermembrane solutions of the above equations.

3. General solutions of the continuum BPS equations

In this section, we like to present the general solution of the BPS equations (2.21).[1] Let us first consider the second BPS equation in (2.21), which may be rewritten as

$$\sqrt{\det h_{\alpha\beta}} = \left(\frac{\mu p^+}{3}\right) Z\,, \tag{3.1}$$

where the induced metric is defined by

$$h_{\alpha\beta} = \frac{\partial Y^\gamma}{\partial \sigma^\alpha}\frac{\partial Y^\gamma}{\partial \sigma^\beta}\,.$$

(The Greek indices run over 1 and 2.) This can always be solved by appropriate choice of the functions $X^\alpha(\sigma)$ for any $Z(\sigma)$. Now noting

$$\begin{aligned}\{Y^\gamma,\{Y^\gamma,Z\}\} &= \epsilon^{\alpha\beta}\frac{\partial Y^\gamma}{\partial\sigma^\alpha}\frac{\partial Y^\delta}{\partial\sigma^\beta}\frac{\partial}{\partial Y^\delta}\left(\epsilon^{\lambda\kappa}\frac{\partial Y^\gamma}{\partial\sigma^\lambda}\frac{\partial Y^\omega}{\partial\sigma^\kappa}\frac{\partial}{\partial Y^\omega}Z\right)\\ &= \left(\frac{\mu p^+}{3}\right)^2 Z\epsilon^{\gamma\delta}\frac{\partial}{\partial Y^\delta}Z\epsilon^{\gamma\omega}\frac{\partial}{\partial Y^\omega}Z = \frac{1}{2}\left(\frac{\mu p^+}{3}\right)^2 Z\frac{\partial}{\partial Y^\gamma}\frac{\partial}{\partial Y^\gamma}Z^2\end{aligned} \tag{3.2}$$

the first BPS equation of (2.21) becomes

$$\frac{\partial}{\partial Y^\gamma}\frac{\partial}{\partial Y^\gamma}Z^2 = -4\,. \tag{3.3}$$

Introducing a complex coordinate $W = Y_1 + iY_2$, the above equation may be rewritten as $\partial_{\overline{W}}\partial_W Z^2 = -1$. Integrating this equation once, we get

$$\partial_W Z^2 = -\overline{W} + f(W)\,, \tag{3.4}$$

where $f(W)$ is an arbitrary function of only W at this stage. One may take $f(W)$ as the following form,

$$f(W) = \sum_k \frac{a_k}{W-b_k} + g(W)\,, \tag{3.5}$$

where $g(W)$ is analytic on the worldvolume of supermembrane. The precise restriction of $G(W)$ will be specified further below. One more integration leads to the solution

$$Z^2 + |W|^2 = A + G(W) + \overline{G(W)} + \sum_k a_k \ln(W-b_k) + \sum_k \bar{a}_k \ln(\overline{W}-\bar{b}_k)\,, \tag{3.6}$$

[1] We like to thank J. Hoppe for pointing out a way to solve the BPS equations.

where $G(W)$ is the integral of $g(W)$ and A is a real constant. To avoid the possibility of singular surfaces, a_k should be taken to be real. Then the final form of the solution reads[2]

$$Z^2 + |W|^2 = A + G(W) + \overline{G(W)} + \sum_{k=1}^{g} a_k \ln|W - b_k|^2 \,. \tag{3.7}$$

As mentioned before, the Gauss law constraint is the integrability condition for the X^- equation. Since we solved the Gauss law constraint, X^- equation in (2.5) may be integrated. This is ture locally but there may remain a global issue[3][4]. To see this, first note that eq. (2.5) may be rearranged to the form

$$4\frac{\partial}{\partial Y^\alpha}X^- = \frac{\mu i}{6}\frac{\partial}{\partial Y^\alpha}\left(G(W) - \overline{G(W)} + \sum_{k=1}^{g} a_k \left[\ln(W - b_k) - \ln(\overline{W} - \bar{b}_k)\right] \right), \tag{3.8}$$

where we used the BPS equations, the solution (3.7) and the fact $\partial G/\partial X_1 = -i\partial G/\partial X_2$. The solution becomes then [4]

$$\frac{X^-}{R} = -\frac{\mu}{12R}\left(C(t) + \mathrm{Im}(G) + \sum_{k=1}^{g} a_k\theta_k \right), \tag{3.9}$$

where the angle variable θ_k is defined by $[\ln(W - b_k) - \ln(\overline{W} - \bar{b}_k)]/(2i)$ and $C(t)$ a function of time only. For the global issue, only the logarithmic parts are relevant. In order to have a surface that is closed in the space including the x^- direction, the coefficient $\alpha_k = \frac{\mu}{12R}a_k$ has to be a rational number. For $g = 1$ case, θ_1 and x^- space has a topology of torus. The above solution represents a curve in this space, which is only closed when α_1 is fractional. This argument may be generalized to the higher genus cases too.

One may wonder what is happening after the matrix regularization. For the finite dimensional matrix configurations, such a topological information will be lost completely and one finds no restriction for the coefficient a_k. For the noncompact infinite dimensional configurations, one may consider the case of supertubes corresponding to $\mu = 0$ limit. The logarithmic behaviors in the above is closely related to membrane configuration pulled out by the fundamental strings [17, 28, 29]. The winding corresponding to the fundamental string charges was found to be fractional after taking care of the quantization of the dynamical modes of supertubes [19]. Further investigation of this issue for the case of the matrix model will be of interest.

Let us now discuss about details of solutions. The choice of $G(W) = a_k = 0$ corresponds to the spherical membrane $Z^2 + |W|^2 = l^2$. The arbitrary deformations of the sphere by introducing $G(W)$ will be also solutions. For instance, the ellipsoid in reference [2] is given with the choice of $G(W) = c\,W^2$ with $|c| < 1$. For small deformation the sphere topology remains. For large deformation, the supermembrane may even become noncompact. This happens for instance choosing $|c| > 1$.

[2]Strictly speaking, (3.7) is the solution of (3.3) with delta-function sources at $W = b_k$. However, the sources are not included into the worldvolume of the membrane, which defines the domains of eq. (3.3).

[3]We like to thank Andrei Mikhailov for bringing this issue to us.

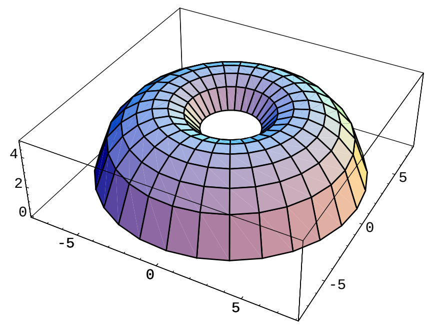

Figure 1: We illustrate the genus one supermembrane drawn for $z \geq 0$.

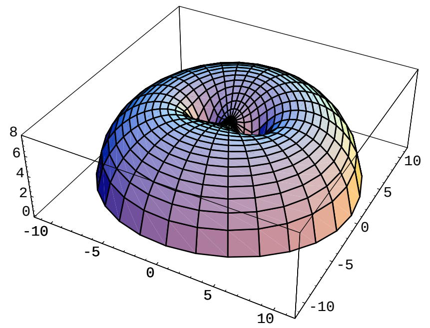

Figure 2: The shape of genus two supermembrane drawn for $z \geq 0$.

The hamiltonian and the angular momentum defined by are conserved and, using the BPS equations, one may show that

$$H = -\frac{\mu}{3}J = \frac{\mu^2 p^+}{9}\int d^2\sigma(Y_1^2 + Y_2^2 - 2Z^2)\,, \tag{3.10}$$

for the compact surfaces. Only noncompact surfaces may carry nonvanishing fundamental string charges. For the sphere, one may easily see that the energy as well as the angular momentum vanish. The supermembrane of ellipsoidal shape has nonvanishing energy and angular momentum.

Let us turn to the case of $g = 1$. Taking $G(W) = 0$, one has

$$Z^2 + |W|^2 - a\ln(\frac{|W|^2}{a}) = l^2 + a\,, \tag{3.11}$$

with $a > 0$. In figure 1, we depict the shape of the genus one surface. When $l = 0$, the surface becomes a circle of radius $\sqrt{a}$ in the $Z = 0$ plane. For finite l, the surface has the shape of doughnut, which is genus one.

All the higher genus solution may be constructed in a similar manner. The positive integer g in (3.7) counts the genus number, i.e. the number of holes. Figure 2 illustrates a genus two surface described by

$$Z^2 + |W|^2 - a\ln|W-b|^2 - a\ln|W+b|^2 = A\,, \tag{3.12}$$

where again $a > 0$ and we take b to be real for simplicity.

Noncompact surfaces are also possible. Whenever a_k is negative, the resulting surface is infinitely extended in the $\pm\, z$ directions at $W = b_k$. Figure 3 shows a simple such case with the rotational symmetry around z-axis; it is described by the equation,

$$Z^2 + |W|^2 + \tilde{a}\ln(|W|^2/\tilde{a}) = l^2 + \tilde{a}\,, \tag{3.13}$$

with $\tilde{a} > 0$. If one introduces many logarithms with negative a_k, the supermembranes has correspondingly many spikes in the $\pm\, z$ directions.

There is again arbitrariness of the shape by changing the analytic function $G(W)$. By an appropriate scaling of the variables in the limit $\mu \to 0$, one may recover the supertube solutions [2, 12, 17] from the one in figure 3 and its deformations. One example of such scaling limit is to take $X_1 \to X_1$, $X_2 \to X_2$, $Z \to Z/\sqrt{\mu}$ as $\mu \to 0$. One can see that the BPS equations for supertubes are recovered in this scaling limit.

Finally, let us discuss the precise restriction on the possible choice of $G(W)$. Let the domain of supermembrane be the projection of the worldvolume of supermembrane to the W plane. Then $G(W)$ should be analytic in the closure of the domain of supermembrane. Singularities of the type $(W-b)^{-n}$ with $n>0$ are not allowed within the closure of the domain in order to avoid singular surfaces.

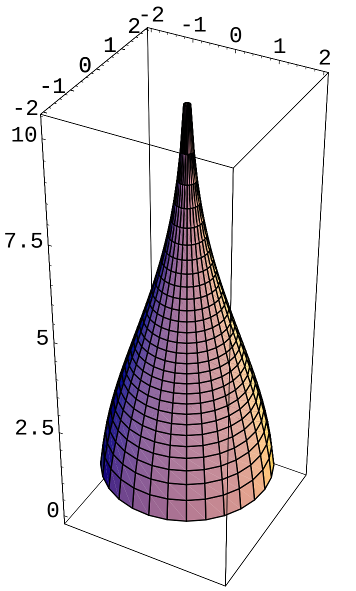

Figure 3: We illustrate a supermembrane of tubular shape for $z \geq 0$.

4. Solutions of matrix BPS equations

In the previous section, we presented the general solution of our continuum BPS equations. We now like to analyze the matrix BPS equations in (2.17). As discussed in reference [4], the matrix BPS equations are solved by the following algebra,

$$\begin{aligned} [W_+, W_-] &= 2\left(\frac{\mu}{3R}\right) Z \\ [Z, W_-] &= -\left(\frac{\mu}{3R}\right) W_- + f(W_+)\,, \end{aligned} \tag{4.1}$$

where $W_+ = Y_1 + iY_2$ is now a matrix and $f(W_+)$ is a function of only W_+. We do not know how to solve this algebra generically. Instead, we like to solve for the case of fuzzy torus, which is described by the above algebra with the choice of

$$f(Y_+) = \left(\frac{\mu}{3R}\right)^3 \frac{a}{Y_+}\,.$$

For the simplicity of the expression, let us introduce the following scaled variables,

$$L_1 = \left(\frac{3R}{\mu}\right) Y_1\,, \qquad L_2 = \left(\frac{3R}{\mu}\right) Y_2\,, \qquad L_3 = \left(\frac{3R}{\mu}\right) Z\,, \tag{4.2}$$

and $L_+ = L_1 + iL_2$.

For the n dimensional representation with L_3 diagonalized, the algebra turns into a set of simple algebraic equations if one takes an ansatz,

$$L_+ = \sum_{i=1}^{n} b_i\, |i\rangle\langle i+1|\,, \tag{4.3}$$

where we define $|n+1\rangle = |1\rangle$. The ansatz here is partly motivated by the consideration of the rotational symmetry.

Using $(L_+)^{-1} = \sum_{i=1}^{n} b_i^{-1} \, |i+1\rangle\langle i|$, the BPS equations become

$$\begin{aligned} 2z_i &= |b_{i-1}|^2 - |b_i|^2 \,, \\ z_{i+1} - z_i &= 1 - a/|b_i|^2 \,, \end{aligned} \tag{4.4}$$

where z_i is the diagonal entry of L_3. Here we use the convention $z_{n+1} = z_1$, $z_0 = z_n$ and $b_0 = b_n$. For $n = 2$, no nontrivial representation with $L_3 \neq 0$. For $L_3 = 0$, one may solve the problem for general n. Simply one has $|b_k| = \sqrt{a}$ for all k. L_+ can be diagonalized with eigenvalues $\sqrt{a} e^{\frac{2\pi k}{n} i}$ with $k = 1, 2, \cdots, n$. This is not the most general solution with $Z = 0$; it may be found by discarding the ansatz (4.3) and directly solving the algebra with $Z = 0$. In the diagonalized basis of L_+, the only condition is that the absolute of the eigenvalues should be $\sqrt{a}$, which agrees with the continuum case. For the solutions, $H = -\mu J/3 = na\mu^2/(9R)$, so they carry a nonvanishing angular momentum.

For $n = 3$, we have just considered the case where all $|b_k|^2$'s are the same. In the remaining cases, at least two of $|b_k|^2$ should be the same. Without loss of generality, we let $|b_1|^2 = |b_2^2|$ using the cyclic property of our ansatz. The solutions are given by

$$\begin{aligned} |b_1|^2 = |b_2|^2 &= 1 \pm \sqrt{1 - 4a/3} \,, \\ |b_3|^2 &= \frac{1 \mp \sqrt{1 - 4a/3}}{2} \,, \end{aligned} \tag{4.5}$$

where a is constrained by $0 \leq a \leq \frac{3}{4}$. The corresponding L_3 is given by

$$L_3 = \begin{bmatrix} -\frac{1 \pm 3\sqrt{1-4a/3}}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1 \pm 3\sqrt{1-4a/3}}{4} \end{bmatrix} \tag{4.6}$$

In this solution, ρ^2 defined by $L_1^2 + L_2^2$ is also diagonal, which reflects the rotational symmetry around z axis. Its expression is given by

$$\rho^2 = \begin{bmatrix} \frac{3 \pm \sqrt{1-4a/3}}{4} & 0 & 0 \\ 0 & 1 \pm \sqrt{1 - 4a/3} & 0 \\ 0 & 0 & \frac{3 \pm \sqrt{1-4a/3}}{4} \end{bmatrix} \tag{4.7}$$

The locations of the eigenvalues in $z - \rho$ plane are depicted in figure 3.

From the figure 4, one can see that the crude feature of genus one surface emerges though the topology here is not well defined due to fuzziness. The hamiltonian for this solution is evaluated as

$$H = -\frac{\mu J}{3} = \frac{\mu^2}{36R} \left(5 + 6a \pm 3\sqrt{1 - 4a/3} \right) . \tag{4.8}$$

Let us now move to the cases of 4×4 matrices. There are three classes of solutions. The first is the one with the same $|b_k|^2$'s for all k and $L_3 = 0$, which is already described in the above. The second is the case where $|b_1|^2 = |b_2|^2$ and $|b_3|^2 = |b_4|^2$ up to any permutation of the entries. The solution of the algebraic equations is found as $|b_1|^2 = |b_2|^2 = 1 \pm \sqrt{1 - a}$

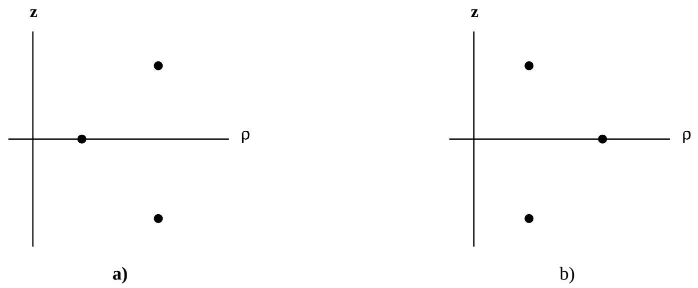

Figure 4: The locations of the eigenvalues in $z - \rho$ plane; a) is for the upper sign or for the lower sign with $0 \leq a \leq 2/3$ and b) for the lower sign with $2/3 \leq a \leq 3/4$.

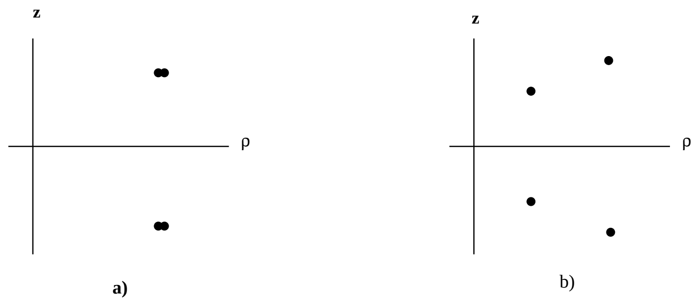

Figure 5: The locations of the eigenvalues in $z - \rho$ plane with 4×4 matrices; a) is for the case where $|b_1|^2 = |b_2|^2$ and $|b_3|^2 = |b_4|^2$ and b) for the case of $|b_1|^2 = |b_3|^2$.

and $|b_3|^2 = |b_4|^2 = 1 \mp \sqrt{1-a}$ where $0 \leq a \leq 1$. The eigenvalue distributions of L_3 and ρ^2 are drawn in figure 5a. The corresponding L_3 is given by

$$L_3 = \begin{bmatrix} \mp\sqrt{1-a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \pm\sqrt{1-a} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.9}$$

and

$$\rho^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 \mp \sqrt{1-a} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \mp \sqrt{1-a} \end{bmatrix} \tag{4.10}$$

The last one is the case where only one pair of entries has the same values, e.g. $|b_1|^2 = |b_3|^2$. In this case, $|b_1|^2 = |b_3|^2 = (3 \pm \sqrt{9-8a})/2$, $|b_2|^2|b_4|^2 = a$ and $|b_2|^2 + |b_4|^2 = (5 \pm \sqrt{9-8a})/2$. One finds then $z_1 = -z_4$ and $z_2 = -z_3$. The eigenvalue distribution is depicted in figure 5b.

In these solutions the parameter a is bounded from above and below. Let us note that, for the case of fuzzy sphere, the size of the sphere is determined once the dimension of representations is given. The above restriction on the parameter a for a given dimension of matrices may be understood in a similar context. The parameter a here is related to the size of the hole of the genus one surface.

One can consider the infinite case N but with discrete matrix case. The 1/4 BPS solution represented by this case cannot be confined in a finite space. The solution becomes simplified when there is a rotational symmetry, and the generalization of the above ansatz leads to a recursion relation between coefficient. This would be the matrix generalization of the supertube in the flat space [11]. As the general characteristics of these solutions have similar to those of the generalized supertube found in the supermembrane solution discussed in the previous section, we will not present the detail of the solution. Finally we like to mention again that the cotinuum limit of the previous section may be recovered by taking the large N limit while keeping X_I/N finite.

5. Conclusion

In this note, we consider the BPS supermembrane configurations in the maximally supersymmetric plane wave background in eleven dimensions. The 1/2 BPS configurations are well known spherical giant gravitons only with p^+ and carries zero angular momentum. For the 1/4 BPS equations, the complete solutions of supermembranes are constructed in full generality. They are characterized by arbitrary number of holes as well as the arbitrary cross sectional shapes. They can be either compact or noncompact. For the matrix regularized version, we constructed several solutions focusing on the case of the fuzzy torus. The 1/4 BPS supermembranes are in three space defined by the nonvanishing four form tensor field strength. Additional angular momentum along other directions by rotating the center of mass reduces the invariant supersymmetry to 1/8 and 1/16.

The construction may used to the counting of the degeneracy of the fixed energy and angular momentum sectors for a given charges as done in reference [19]. The construction of the corresponding black hole solutions will be very interesting. Also the characteristics of the less supersymmetric supermembrane suggests the direction to pursue for similar BPS configurations in the $AdS^7 \times S^4$ or $AdS^4 \times S^7$.

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